Axisymmetric Stress Fields in Involute Bodies of Revolution

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A new theory is proposed to define the elastic response of axisymmetric, but otherwise arbitrarily anisotropic and heterogeneous, bodies of revolution. The theory is based upon the application of Reissner's variational principle and an assumed stress field to given subregions of the body and approaches the theory of elasticity in the limit of vanishing subregion thickness. The new theory is particularly appropriate in the treatment of involute bodies, and it includes the effects of pore pressure, free expansion strains, body forces, and temperature-dependent properties.

Introduction

IN this work a theory will be derived to describe the elastic response of a body of revolution, such as a shell-like structure, under axisymmetric loading and/or environmental conditions. The elastic stiffness matrix is not restricted by any special symmetry requirements, aside from the axisymmetric architecture. The principal motivation for this work is the representation of the stress field in an involute body,[1,2] although other forms of composite construction, such as tape-wrapping, filament winding, and lamination, which all have well-defined fiber orientation distributions, can also be described. The defining characteristic of these composite structures is that the stiffness matrix does not depend on θ (but may depend on r and z).

Involute exit cones are important components of rocket nozzles used for military and space applications. Their loading conditions during propellant burn and processing involve high temperatures and possible gas diffusion with the associated pore pressure. Existing analyses of involute bodies are quite limited. The case of cylindrical involutes has been reported in Ref. 3. Contemporary models for involute exit cones are based upon the finite element method, such as the Patches III treatment.[4] Theories of the type given by Reissner and Wan[5,6] represent thin-shell approximations and are sufficiently general to include the class of anisotropy featured here. For thin bodies, one can anticipate close agreement between the results of Refs. 5 and 6 and those of the present model.

The theory developed in the present work follows the approach outlined in Ref. 7 and is based upon a variational theorem derived by Reissner.[8] Although the work is based upon an assumed form of the stress field, an inherent feature, as in Ref. 7, is the capability to improve solution accuracy by increasing the number of subregions. At the limit of vanishing subregion thickness, the results, using the present theory, approach those of heterogeneous elasticity theory.

An example problem of involute cylindrical body will also be presented and results compared to a previous elasticity solution.[3]

Nomenclature and Approach

In this section the stress field developed within a class of elastic bodies of revolution representative of rocket nozzle exit cones and other axisymmetric structures shall be considered. The body is generated by rotating the cross section shown in Fig. 1 about the z axis. Hence, the body is bounded by end planes $z=z_1$, z_2 and inner and outer radii, $r_I(z)$ and $r_0(z)$, respectively. The elastic properties are only restricted by the assumption of axisymmetry; thus, while the elastic stiffness matrix depends only on r and z, it may contain 21 nonzero constants at a given point in space. The surfaces $r=r_I$, r_0 and the end planes $z=z_1$, z_2 may be subjected to traction and/or displacement boundary conditions which are independent of θ. Body forces induced by rotation about the z axis, with constant angular velocity ω as well as gravitational constant g, act throughout the body. Strains caused by free expansion under a temperature change or moisture absorption e_i will be introduced. Pore pressure p, which is motivated by a desire to examine the in-process response of composite bodies, will also be introduced. It should be noted that no stress components are neglected in this formulation so torsional response is present. It will also be convenient to allow the stiffness matrix to depend upon a variable T, which may represent temperature or simply an arbitrary parameter to provide a means of prescribing a given spatial distribution of the stiffness (and free expansion) matrix. Included in the generic class of material architecture represented in this work are bodies formed by tape-wrapping, filament winding, involute construction, and lamination. Because of the need to satisfy traction and displacement continuity conditions in some of these schemes, and also to establish the capability to provide fine subdivisions to improve solution accuracy as in Ref. 7, the formulation will be based upon Reissner's variational principle.[8]

Reissner has shown that the governing equations of elasticity can be obtained as a consequence of the variational equation

$$\delta J=0 \tag{1}$$

where

$$J=\int_V F\mathrm{d}V-\int_{S'}\bar{\tau}_i\xi_i\mathrm{d}S-\int_V B_i\xi_i\mathrm{d}V \tag{2}$$

and

$$F=\tfrac{1}{2}\sigma_{ij}(\xi_{i,j}+\xi_{j,i})-W \tag{3}$$

in terms of cartesian coordinates x_i $(i=1,2,3)$. In these equations, W is the strain energy density expressed in terms of the stresses σ_{ij}; V is the volume; S the entire surface; $\bar{\tau}_i$ the prescribed tractions; B_i the components of body force, ξ_i the displacement components; and S' is the portion of the boundary on which one or more traction components are prescribed. It is understood that both stresses and displacements are subjected to variation in the application of Eq. (1), and summation over the range of repeated subscripts (but not superscripts) is implied throughout this work.

Development of Theory

First an arbitrary region within the body defined by the radii $r_1(z)$ and $r_2(z)$ is considered, as shown in Fig. 1. Let the cross sectional area of this region be denoted by A, and its perimeter

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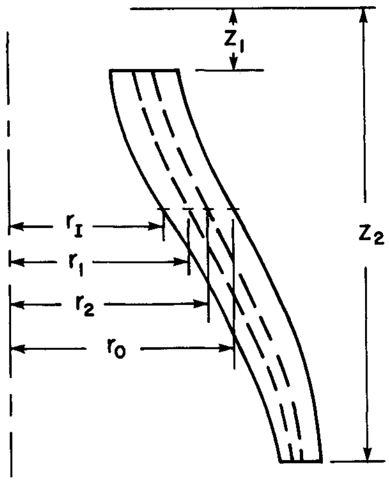

Fig. 1 Geometric configuration of a body of revolution.

by L. The portion of L on which one or more traction components are presecribed will be denoted by L'. Note that the boundary L will include r_I if $r_1 = r_I$ and r_0 if $r_2 = r_0$. A right-handed cylindrical coordinate system (z,θ,r) is introduced and contracted notation is employed in the representation of the stress and strain components, i.e.,

$$\sigma_1 = \sigma_z, \sigma_2 = \sigma_\theta, \sigma_3 = \sigma_r, \sigma_4 = \sigma_{r\theta}, \sigma_5 = \sigma_{zr}, \sigma_6 = \sigma_{z\theta} \quad (4)$$

and the analogous relation for the engineering strain components ϵ_i $(i = 1\text{-}6)$. The r, θ, z components of displacement are designated as u, v, w, respectively. Finally, the function J for the arbitrary region is denoted by J_p, where J is defined by Eq. (2).

The form of the stress components is now assumed within arbitrary region A and is given by

$$\sigma_i = \sigma_{iJ} f_J^{(i)} \qquad (i = 1\text{-}6,\ J = 1\text{-}5) \quad (5)$$

where σ_{iJ} are functions of z only and

$$f_1^{(1)} = f_1^{(2)} = f_1^{(3)} = f_1^{(6)} = (r_2 - r)/(r_2 - r_1)$$

$$f_2^{(1)} = f_2^{(2)} = f_2^{(3)} = f_2^{(6)} = (r - r_1)/(r_2 - r_1)$$

$$f_1^{(4)} = f_1^{(5)} = \frac{r}{r_1} f_1^{(1)}$$

$$f_2^{(4)} = f_2^{(5)} = \frac{r}{r_2} f_2^{(1)}$$

$$f_3^{(3)} = r^3 - (r_1^2 + r_1 r_2 + r_2^2) r + r_1 r_2 (r_1 + r_2)$$

$$f_4^{(3)} = r^2 - (r_1 + r_2) r + r_1 r_2$$

$$f_5^{(3)} = \frac{f_4^{(3)}}{r_1 r_2 r}$$

$$f_3^{(4)} = \frac{(r_1^2 + r_1 r_2 + r_2^2) r^2 - (r_1 + r_2)(r_1^2 + r_2^2) r}{r_1^3 r_2^3} + \frac{1}{r^2}$$

$$f_3^{(5)} = \frac{(r_1 + r_2) r^2 - (r_1^2 + r_1 r_2 + r_2^2) r}{r_1^2 r_2^2} + \frac{1}{r} \quad (6)$$

with

$$\sigma_{iJ} = f_J^{(i)} = 0 \ (i = 1,2,6 \text{ and } J = 3,4,5;\ J = 4,5 \text{ and } i = 4,5) \quad (7)$$

arise by assuming that σ_1, σ_2, and σ_6 are linear functions of r in the region A and then determining the form of the remaining stress components from the equations of equilibrium in axisymmetric elasticity. Furthermore, the functions σ_{iJ} are chosen to satisfy the following relations

$$\sigma_{i\alpha}(z) = \sigma_i(r_\alpha, z) \qquad (i = 1\text{-}6,\ \alpha = 1,2) \quad (8)$$

In general, the strain energy density of an elastic anisotropic body, including the influence of free expansion and pore pressure,[9,10] is given by

$$W = 1/2 S_{ij}\sigma_i\sigma_j + \sigma_i e_i - S_{ij}\sigma_i\sigma_j^{(0)} \quad (9)$$

where S_{ij} is the compliance matrix; e_i represents the engineering free expansion strain components; and

$$\sigma_i^{(0)} = \begin{cases} -c_i p & (i = 1,2,3) \\ 0 & (i = 4,5,6) \end{cases} \quad (10)$$

Here, p is the pore pressure and c_i are parameters that depend on the shape and size of the region of open porosity. Also, σ_i is the total stress acting on the bulk material, or the sum of the stress on the solid material and the pore pressure stress $\sigma_i^{(0)}$. It has been assumed that the pore pressure p is a known function in this formulation and that the porsity is unaffected by the deformation.[10]

In other words, the functions $f_J^{(i)}$ not displayed in Eq. (6) and the corresponding σ_{iJ} all vanish. The relations in Eqs. 5 and 6

In the subsequent derivation of the governing equations (see Ref. 7 also), the integrations will give rise to weighted average displacements and displacements on the surfaces $r = r_1$, r_2. Therefore, these definitions are made

$$(\bar{q}, q^*, \hat{q}, \bar{\bar{q}}) = \int_{r_1}^{r_2} q(1, r, r^2, r^3)\,\mathrm{d}r \quad (11)$$

where q may represent either u, v, or w. Also let

$$q_\alpha(z) = q(r_\alpha, z) \quad (\alpha = 1,2) \quad (12)$$

with the same interpretation of q. It will also be necessary to apply Leibnitz's theorem in the form

$$\int_{r_1(z)}^{r_2(z)} \frac{\partial}{\partial z} F(r,z)\,\mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}z}\int_{r_1(z)}^{r_2(z)} F(r,z)\,\mathrm{d}r - r_2' F(r_2,z) + r_1' F(r_1,z) \quad (13)$$

where primes represent derivatives with respect to z.

We now substitute Eqs. (3-12) into Eq. (2), using Eq. (13) where appropriate and take the first variation, which leads to the result

$$\begin{aligned} \frac{\partial J}{2\pi} = \int_{z_1}^{z_2} &\{ (\mu_{iJ} + \chi_{iJ})\delta\sigma_{iJ} + [(\sigma_{32} - r_2'\sigma_{52})\delta u_2 \\ &+ (\sigma_{42} - r_2'\sigma_{62})\delta v_2 + (\sigma_{52} - r_2'\sigma_{12})\delta w_2] r_2 \\ &- [(\sigma_{31} - r_1'\sigma_{51})\delta u_1 + (\sigma_{41} - r_1'\sigma_{61})\delta v_1 \\ &+ (\sigma_{51} - r_1'\sigma_{11})\delta w_1] r_1 - (F_1\delta\bar{u} + F_2\delta u^* \\ &+ F_3\delta\hat{u} + F_4\delta\bar{\bar{u}} + F_5\delta v^* + F_6\delta\hat{v} \\ &+ F_7\delta w^* + F_8\delta\hat{w})\}\mathrm{d}z + \left\{ \left[\frac{r_1\sigma_{52} - r_2\sigma_{51}}{r_1 r_2 (r_2 - r_1)} + \frac{(r_1 + r_2)\sigma_{53}}{r_1^2 r_2^2} \right] \delta\bar{u} \right. \\ &+ \left[\frac{r_2^2\sigma_{51} - r_1^2\sigma_{52}}{r_1 r_2 (r_2 - r_1)} - \frac{(r_1^2 + r_1 r_2 + r_2^2)\sigma_{53}}{r_1^2 r_2^2} \right] \delta\hat{u} + \sigma_{53}\delta\bar{\bar{u}} \\ &+ \frac{(r_2\sigma_{61} - r_1\sigma_{62})}{r_2 - r_1}\delta v^* + \frac{(\sigma_{62} - \sigma_{61})}{r_2 - r_1}\delta\hat{v} \\ &\left. + \frac{(r_2\sigma_{11} - r_1\sigma_{12})}{r_2 - r_1}\delta w^* + \frac{(\sigma_{12} - \sigma_{11})}{r_2 - r_1}\delta\hat{w} \right\} \Bigg|_{z = z_1}^{z = z_2} \\ &- \int_{L'} (\bar{\tau}_r\delta u + \bar{\tau}_\theta\delta v + \bar{\tau}_z\delta w) r\,\mathrm{d}L \end{aligned} \quad (14)$$

where

$$\chi_{iJ}=\eta_{iJ}+P_{iJ}-E_{iJ}-\hat{S}_{ijKJ}\sigma_{jK} \qquad (i,j=1\text{-}6;\ J,K=1\text{-}5) \tag{15}$$

$$P_{iJ}=\int_{r_1}^{r_2} S_{ij}\sigma_j^{(0)} f_J^{(i)}\, r\mathrm{d}r \qquad '' \tag{16}$$

$$E_{iJ}=\int_{r_1}^{r_2} e_i f_J^{(i)}\, r\mathrm{d}r \qquad '' \tag{17}$$

$$\hat{S}_{ijKJ}=\int_{r_1}^{r_2} S_{ij} f_K^{(j)} f_J^{(i)}\, r\mathrm{d}r \qquad '' \tag{18}$$

Also

$$\eta_{11}=\frac{r_2 w^{*\prime}-\hat{w}'}{r_2-r_1} \qquad \eta_{41}=\frac{4\hat{v}-3r_2 v^*}{r_1(r_2-r_1)}$$

$$\eta_{12}=\frac{\hat{w}'-r_1 w^{*\prime}}{r_2-r_1} \qquad \eta_{42}=\frac{3r_1 v^*-4\hat{v}}{r_2(r_2-r_1)}$$

$$\eta_{21}=\frac{r_2\bar{u}-u^*}{r_2-r_2}$$

$$\eta_{43}=\frac{3(r_1^3+r_2r_1^2+r_1r_2^2+r_2^3)v^*-4(r_1^2+r_1r_2+r_2^2)\hat{v}}{r_1^3r_2^3}$$

$$\eta_{22}=\frac{u^*-r_1\bar{u}}{r_2-r_1} \qquad \eta_{51}=\frac{3\hat{w}-2r_2w^*+r_2\hat{u}'-\bar{\bar{u}}'}{r_1(r_2-r_1)}$$

$$\eta_{31}=\frac{2u^*-r_2\bar{u}}{r_2-r_1} \qquad \eta_{52}=\frac{2r_1w^*-3\hat{w}+\bar{\bar{u}}'-r_1\hat{u}'}{r_2(r_2-r_1)}$$

$$\eta_{32}=\frac{r_1\bar{u}-2u^*}{r_2-r_1}$$

$$\eta_{53}=\frac{(r_1^2+r_1r_2+r_2^2)(2w^*-\hat{u}')+(r_1+r_2)(\bar{\bar{u}}'-3\hat{w})+r_1^2r_2^2\bar{u}'}{r_1^2r_2^2}$$

$$\eta_{33}=-r_1r_2(r_1+r_2)\bar{u}+2(r_1^2+r_1r_2+r_2^2)u^*-4\bar{\bar{u}}$$

$$\eta_{61}=\frac{r_2v^{*\prime}-\hat{v}'}{r_2-r_1}$$

$$\eta_{34}=-r_1r_2\bar{u}+2(r_1+r_2)u^*-3\hat{u}$$

$$\eta_{35}=\frac{(r_1+r_2)\bar{u}-2u^*}{r_1r_2} \qquad \eta_{62}=\frac{\hat{v}'-r_1v^{*\prime}}{r_2-r_1} \tag{19}$$

with

$$\eta_{iJ}=0\ (i=1,2,6 \text{ and } J=3,4,5;\ i=4,5 \text{ and } J=4,5) \tag{20}$$

and

$$\mu_{11}=r_1r_1'w_1 \qquad \mu_{42}=r_2v_2$$

$$\mu_{12}=-r_2r_2'w_2 \qquad \mu_{51}=r_1r_1'u_1-r_1w_1$$

$$\mu_{31}=-r_1u_1 \qquad \mu_{52}=r_2w_2-r_2r_2'u_2$$

$$\mu_{32}=r_2u_2 \qquad \mu_{61}=r_1r_1'v_1$$

$$\mu_{41}=-r_1v_1 \qquad \mu_{62}=-r_2r_2'v_2 \tag{21}$$

with

$$\mu_{iJ}=0\ (i=1\text{-}6 \text{ and } J=3\text{-}5);\ i=2 \text{ and } J=1,2) \tag{22}$$

In general, the integrals of Eqs. (16-18) cannot be expressed in closed form representation, but can be easily evaluated by numerical means. Finally, we have employed the following contractions in Eq. (14),

$$F_1=\frac{r_2\sigma_{31}-r_1\sigma_{32}}{r_2-r_1}+r_1r_2(r_1+r_2)\sigma_{33}+r_1r_2\sigma_{34}$$
$$-\frac{(r_1+r_2)\sigma_{35}}{r_1r_2}+\frac{r_1\sigma_{22}-r_2\sigma_{21}}{r_2-r_1}+\sigma_{53}'$$

$$F_2=\frac{2(\sigma_{32}-\sigma_{31})}{r_2-r_1}-2(r_1^2+r_1r_2+r_2^2)\sigma_{33}-2(r_1+r_2)\sigma_{34}$$
$$+\frac{2\sigma_{35}}{r_1r_2}+\frac{\sigma_{21}+\sigma_{22}}{r_2-r_1}$$

$$F_3=3\sigma_{34}+\frac{r_2^2\sigma_{51}'-r_1^2\sigma_{52}'}{r_1r_2(r_2-r_1)}-\frac{(r_1^2+r_1r_2+r_2^2)\sigma_{53}'}{r_1^2r_2^2}$$
$$+\left[\frac{(2r_1-r_2)r_2r_1'-r_1^2r_2'}{r_1^2(r_2-r_1)^2}\right]\sigma_{51}+\left[\frac{(2r_2r_1)r_1r_2'-r_2^2r_1'}{r_2^2(r_2-r_1)^2}\right]\sigma_{52}$$
$$+\left[\frac{r_2^2r_1'(r_1+2r_2)+r_1^2r_2'(r_2+2r_1)}{r_1^3r_2^3}\right]\sigma_{53}+\rho\omega^2$$

$$F_4=4\sigma_{33}+\frac{r_1\sigma_{52}'-r_2\sigma_{51}'}{r_1r_2(r_2-r_1)}+\frac{(r_1+r_2)\sigma_{53}'}{r_1^2r_2^2}$$
$$+\left[\frac{r_1r_2'-(2r_1-r_2)r_1'}{r_1^2(r_2-r_1)^2}\right]\sigma_{51}+\left[\frac{r_2r_1'-(2r_2-r_1)r_2'}{r_2^2(r_2-r_1)^2}\right]\sigma_{52}$$
$$-\left[\frac{(r_1+2r_2)r_2r_1'+(r_2+2r_1)r_1r_2'}{r_1^3r_2^3}\right]\sigma_{53}$$

$$F_5=\frac{3r_2\sigma_{41}}{r_1(r_2-r_1)}-\frac{3r_1\sigma_{42}}{r_2(r_2-r_1)}-\frac{3(r_1^3+r_2r_1^2+r_1r_2^2+r_2^3)}{r_1^3r_2^3}\sigma_{43}$$
$$+\frac{r_2\sigma_{61}'-r_1\sigma_{62}'}{r_2-r_1}+\frac{(r_2r_1'-r_1r_2')(\sigma_{61}-\sigma_{62})}{(r_2-r_1)^2}$$

$$F_6=\frac{-4\sigma_{41}}{r_1(r_2-r_1)}+\frac{4\sigma_{42}}{r_2(r_2-r_1)}+\frac{4(r_1^2+r_1r_2+r_2^2)\sigma_{43}}{r_1^3r_2^3}$$
$$+\frac{\sigma_{62}'-\sigma_{61}'}{r_2-r_1}+\frac{(r_2'-r_1')(\sigma_{61}-\sigma_{62})}{(r_2-r_1)^2}$$

$$F_7=\frac{r_2\sigma_{11}'-r_1\sigma_{12}'}{r_2-r_1}+\frac{(r_2r_1'-r_1r_2')(\sigma_{11}-\sigma_{12})}{(r_2-r_1)^2}$$
$$+\frac{2(r_2^2\sigma_{51}-r_1^2\sigma_{52})}{r_1r_2(r_2-r_1)}-\frac{2(r_1^2+r_1r_2+r_2^2)\sigma_{53}}{r_1^2r_2^2}+\rho g$$

$$F_8=\frac{\sigma_{12}'-\sigma_{11}'}{r_2-r_1}+\frac{(r_2'-r_1')(\sigma_{11}-\sigma_{12})}{(r_2-r_1)^2}$$
$$+\frac{3(r_1\sigma_{52}-r_2\sigma_{51})}{r_1r_2(r_2-r_1)}+\frac{3(r_1+r_2)\sigma_{53}}{r_1^2r_2^2} \tag{23}$$

where ρ is the mass density, ω the angular velocity, and g the acceleration due to gravity.

Equation (14) expresses a general relation valid for any arbitrary region or strip ($r_1 \le r \le r_2$, $z_1 \le z \le z_2$). Consequently, Eq. (14) can be utilized to define δJ for the entire body by (conceptually) introducing N strips to fill area A. Taking account of interrelations which must exist among the variational quantities and applying Eq. (1) then, the following field equations *for each strip* ($r_1 \le r \le r_2$) result

$$F_\alpha=0\,(\alpha=1\text{-}8) \tag{24}$$

and

$$\chi_{21}=\chi_{22}=\chi_{33}=\chi_{34}=\chi_{35}=\chi_{43}=\chi_{53}=0 \tag{25}$$

while for the strip with $r_1=r_I$, we have

$$\chi_{11}+\chi_{51}r_1'+\chi_{31}(r_1')^2=0 \tag{26}$$

$$\chi_{61}+\chi_{41}r_1'=0 \tag{27}$$

and for the strip with $r_2=r_0$, we have

$$\chi_{12}+\chi_{52}r_2'+\chi_{32}(r_2')^2=0 \tag{28}$$

$$\chi_{62}+\chi_{42}r_2'=0 \tag{29}$$

In the event that more than one strip is present, continuity† of traction and displacement between adjacent strips with an interface at $r=r_k$ requires that

$$\chi_{i2}^{(k-1)}+\chi_{i1}^{(k)}=0 \qquad (i=1,3,4,5,6) \tag{30}$$

$$\chi_{12}^{(k-1)}+\chi_{52}^{(k-1)}r_k'+\chi_{32}^{(k-1)}(r_k')^2=0 \tag{31}$$

$$\chi_{62}^{(k-1)}+\chi_{42}^{(k-1)}r_k'=0 \tag{32}$$

$$\sigma_{32}^{(k-1)}-\sigma_{31}^{(k)}+[\sigma_{51}^{(k)}-\sigma_{52}^{(k-1)}]r_k'=0 \tag{33}$$

$$\sigma_{42}^{(k-1)}-\sigma_{41}^{(k)}+[\sigma_{61}^{(k)}-\sigma_{62}^{(k-1)}]r_k'=0 \tag{34}$$

$$\sigma_{52}^{(k-1)}-\sigma_{51}^{(k)}+[\sigma_{11}^{(k)}-\sigma_{12}^{(k-1)}]r_k'=0 \tag{35}$$

where the strip superscript (k) has been introduced such that k increases in the direction of increasing r. On the boundary $r=r_I$, tractions and/or displacements may be prescribed according to

$$\begin{aligned}
&\chi_{31}=r_I\tilde{u}_{II} \text{ or } \sigma_{51}\sin\gamma_I-\sigma_{31}\cos\gamma_I=\tilde{\tau}_{rI}\\
&\chi_{41}=r_I\tilde{v}_I \text{ or } \sigma_{61}\sin\gamma_I-\sigma_{41}\cos\gamma_I=\tilde{\tau}_{\theta I}\\
&\chi_{31}r_I'+\chi_{51}=r_I\tilde{w}_I \text{ or } \sigma_{11}\sin\gamma_I-\sigma_{51}\cos\gamma_I=\tilde{\tau}_{zI}
\end{aligned} \tag{36}$$

where

$$\tan\gamma_I=r_I' \tag{37}$$

Similarly, on $r=r_0$ we may prescribe

$$\begin{aligned}
&\chi_{32}=-r_0\tilde{u}_0 \text{ or } \sigma_{32}\cos\gamma_0-\sigma_{52}\sin\gamma_0=\tilde{\tau}_{r0}\\
&\chi_{42}=-r_0\tilde{v}_0 \text{ or } \sigma_{42}\cos\gamma_0-\sigma_{62}\sin\gamma_0=\tilde{\tau}_{\theta 0}\\
&\chi_{32}r_0'+\chi_{52}=-r_0\tilde{w}_0 \text{ or } \sigma_{52}\cos\gamma_0-\sigma_{12}\sin\gamma_0=\tilde{\tau}_{z0}
\end{aligned} \tag{38}$$

where

$$\tan\gamma_0=r_0' \tag{39}$$

and all functions containing tildes have prescribed values. All of the dependent variables in Eq. (36) correspond to $k=1$, while those in Eq. (38) correspond to $k=N$.

Finally, the boundary conditions on each end $z=z_1,z_2$ require that one term‡ from each of the following products must be prescribed *for each strip*

$$\begin{aligned}
&\left[\frac{r_1\sigma_{52}-r_2\sigma_{51}}{r_1r_2(r_2-r_1)}+\frac{(r_1+r_2)\sigma_{53}}{r_1^2r_2^2}\right]\bar{\bar{u}},\\
&\left[\frac{r_2^2\sigma_{51}-r_1^2\sigma_{52}}{r_1r_2(r_2-r_1)}-\frac{(r_1^2+r_1r_2+r_2^2)\sigma_{53}}{r_1^2r_2^2}\right]\hat{u},\\
&\sigma_{53}\bar{u},\ (r_2\sigma_{61}-r_1\sigma_{62})v^*, \qquad (\sigma_{62}-\sigma_{61})\hat{v},\\
&(r_2\sigma_{11}-r_1\sigma_{12})w^*, \qquad (\sigma_{12}-\sigma_{11})\hat{w}
\end{aligned} \tag{40}$$

and for internal consistency in the theory, prescribed traction components corresponding to σ_1, σ_5, and σ_6 on the appropriate portions of the end planes must be distributed according to Eq. (5).

This completes the formulation of the present theory. We note that the field equations plus lateral surface boundary conditions, Eqs. (24-36) and (38), constitute a system of $25N$ equations in terms of a like number of unknowns. Inspection of the governing equations reveals that the interfacial displacements q_α will only appear in the form of *prescribed* functions, hence they are not to be considered as dependent variables. From Eq. (40), $7N$ boundary conditions are required at each end $z=z_1$, z_2.

Rigid Body Displacement

Freedom to prescribe arbitrary sets of traction boundary conditions in boundary value problems is restricted owing to the need to constrain rigid body displacement. Therefore, the general form of rigid body displacement components shall now be derived in the present theory. The case $N=1$ shall be treated specifically, although it can be shown that the results established here are perfectly general.

Consider the traction boundary value problem in which body forces, free expansion strains, and pore pressure are all absent. Furthermore, let σ_{iJ} ($i=1$-6, $J=1$-5) vanish identically. Therefore, it is clear that Eq. (24) and the right-hand members of Eq. (36) and Eq. (38) are identically satisfied, while Eq. (25), in conjunction with Eq. (19), yield $\bar{u}=u^*=\hat{u}=\bar{\bar{u}}=0$ as well as

$$\hat{v}_R=Q_1v_R^*, \qquad \hat{w}_R=Q_2w_R^* \tag{41}$$

where the subscript R stands for rigid body displacement and

$$Q_1=\frac{3(r_I^3+r_0r_I^2+r_Ir_0^2+r_0^3)}{4(r_I^2+r_Ir_0+r_0^2)}, \qquad Q_2=\frac{2(r_I^2+r_Ir_0+r_0^2)}{3(r_I+r_0)} \tag{42}$$

Substituting Eq. (41) into Eqs. (26-29) and carrying out the cumbersome algebraic and differential operations, we observe that the latter relations are all identically satisfied. Hence the rigid body displacement components are correctly given by Eq. (41) and prescription of one of $(\hat{v},v^*)$ and one of $(\hat{w},w^*)$ must accompany (and replace two of) the prescribed traction boundary conditions in traction boundary value problems.

Overall Equilibrium Relations

The reduction (by two) in the number of traction boundary conditions caused by the presence of rigid body displacement does not imply a loss of essential freedom in traction boundary value problems. For, precisely two relations can be shown to exist among the surface tractions as a consequence of satisfying local equilibrium relations, i.e., Eq. (24), everywhere in the region. Physically, these relations correspond to force and moment equilibrium with respect to the z-axis. We shall again demonstrate for the $N=1$ case.

The force equilibrium result is derived by solving the third of the right-hand members of Eq. (36) and Eq. (38) for σ_{51} and σ_{52}, respectively. In turn, these relations are substituted into

†Other interface conditions are possible, but they are not discussed here.

‡As usual, the choice of boundary conditions is not completely arbitrary since rigid body displacement components, as discussed in the next section, are present in this theory.

Eq. (24) with $\alpha = 7,8$. Next, σ_{53} is eliminated from the resulting two equations. This procedure leads to the relation

$$\Delta[\sigma_{11}(r_0^2 + r_I r_0 - 2r_I^2) + \sigma_{12}(2r_0^2 - r_I r_0 - r_I^2)] + 3\int_{z_1}^{z_2}\left[2\left(\frac{\bar{\tau}_{zI} r_I}{\cos\gamma_I} + \frac{\bar{\tau}_{z0} r_0}{\cos\gamma_0}\right) + (r_0^2 - r_I^2)\rho g\right]\mathrm{d}z = 0 \quad (43)$$

where Δ signifies the change in a function from z_1 to z_2.

The moment equilibrium result is derived in a similar fashion, using the second of the right-hand members of Eqs. (36) and (38) in addition to Eq. (24) with $\alpha = 5,6$. In this case, we get

$$\Delta[\sigma_{61}(r_0^3 + r_I r_0^2 + r_0 r_I^2 - 3r_I^3) + \sigma_{62}(3r_0^3 - r_I r_0^2 - r_0 r_I^2 - r_I^3)] + 12\int_{z_1}^{z_2}\left(\frac{\bar{\tau}_{\theta I} r_I^2}{\cos\gamma_I} + \frac{\bar{\tau}_{\theta 0} r_0^2}{\cos\gamma_0}\right)\mathrm{d}z = 0 \quad (44)$$

Hence, the boundary data in traction boundary value problems cannot be chosen arbitrarily since Eqs. (43) and (44) are implied by Eqs. (24), (36), and (38). It is also clear that certain mixed boundary value problems may require satisfaction of only one of the relations Eq. (43) and (44) since the boundary data in the two equations is uncoupled.

Involute Geometry

Involute bodies are formed by laminating identical fabric-reinforced composite plies of uniform thickness in such a way that each ply extends to the extremities of the body in both the radial and axial directions. Various forms of involute construction are prevalent in the fabrication of these bodies. The geometry associated with each of these approaches is described in Ref. 2 in full detail. The present theory is appropriate to model all of these approaches. It is only necessary to define the spatial distribution of the orientation vectors which give the local warp, fill, and normal directions. The parameters used to describe these orientation vectors are the angles α, γ, ψ, and ϕ shown in Fig. 2. The most convenient way to establish the orientation vectors and the associated transformed compliance and free expansion matrices is to compute the direction cosines in terms of these parameters and then apply the standard transformation equations. Here we shall delineate these direction cosine relations.

Let the warp, fill, and normal directions at any point in space be denoted by x_1, x_2, and x_3, respectively, and the positive directions of z, θ, and r be represented by x_1', x_2', and x_3', respectively. Further, let the direction cosines a_{ij} be defined by

$$a_{ij} = \cos(x_i, x_j') \quad (i,j = 1,2,3) \quad (45)$$

Hence, the direction cosines are given by

$$a_{11} = \cos\gamma/\sin\psi\,(\cos\phi\sin\psi - \cos\psi\sin\phi)$$

$$a_{12} = \cos\alpha\sin\phi/\sin\psi$$

$$a_{13} = 1/\sin\psi\,(\sin\gamma\cos\phi\sin\psi - \sin\alpha\sin\phi\cos^2\gamma)$$

$$a_{21} = \cos\gamma/\sin\psi(\sin\alpha\sin\gamma\cos\phi - \sin\phi\sin\psi)$$

$$a_{22} = \cos\alpha\cos\phi/\sin\psi$$

$$a_{23} = -1/\sin\psi(\sin\gamma\sin\phi\sin\psi + \sin\alpha\cos\phi\cos^2\gamma)$$

$$a_{31} = -\cos\alpha\sin\gamma/\sin\psi$$

$$a_{32} = \sin\alpha\cos\gamma/\sin\psi$$

$$a_{33} = \cos\alpha\cos\gamma/\sin\psi \quad (46)$$

The direction cosines are, in general, functions of r and z. The compliance and free expansion strain tensors in the $z\theta r$ coordinate system are now determined from the (known) components in the material symmetry axes (x_i) via the standard transformation equations.

Illustrative Problem

As an example problem the involute cylinder studied in Ref. 3 (Fig. 3), is treated where the engineering moduli in the material symmetry system (x_i) are given by

$$E_{11} = 20\times10^6 \text{ psi}, \quad E_{22} = E_{33} = 1.4\times10^6 \text{ psi}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = .25, \quad G_{12} = G_{13} = 7\times10^5 \text{ psi},$$

$$G_{23} = 5.6\times10^5 \text{ psi}$$

where E and G stand for Young's modulus and engineering shear modulus, respectively, and ν_{ij} is the Poisson ratio measuring strain in the x_j direction under uniaxial stress in the x_i direction. The cylinder is under internal pressure of 1000 psi and the outer surface is traction-free. The inner and outer radii are 3 inches and 4 inches, respectively, and the free expansion strains are zero. The angles α and ϕ at the outer surface are $\pi/12$ and $\pi/3$,, respectively. In the class of problems

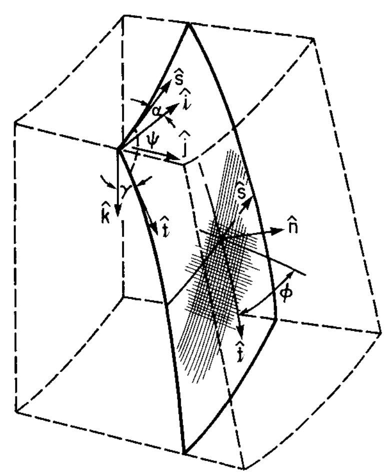

Fig. 2 Involute surface geometry.

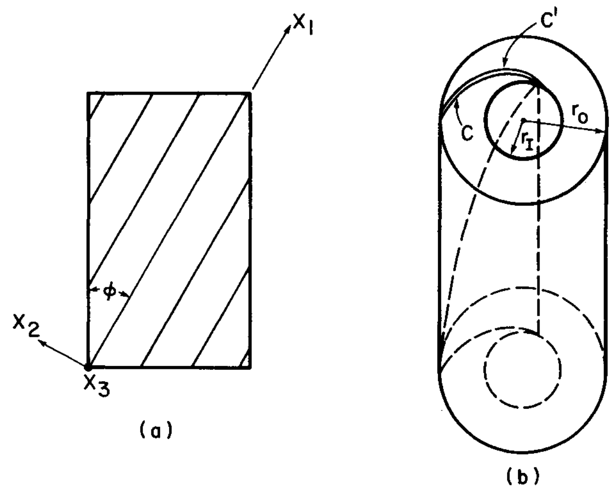

Fig. 3 Involute cylinder configuration.

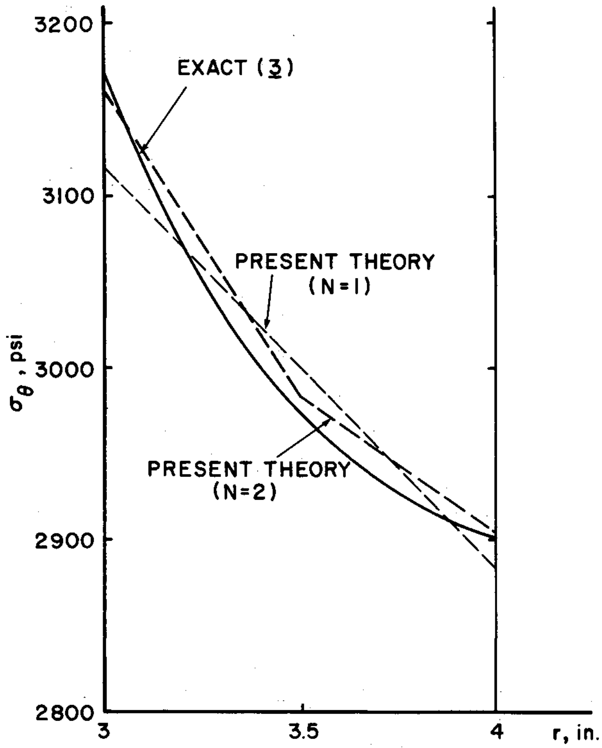

Fig. 4 Hoop stress distribution vs r.

treated in Ref. 3, the stress field was assumed to be independent of axial coordinate z. Therefore, in the present theory, the "stresses" σ_{iJ} are constant. If we apply the end tractions from the solution of Ref. 3, the hoop stress distribution given by the present model for $N=1$ and 2 is shown in Fig. 4 where the solution from Ref. 3 is also given.

Conclusion

An approximate theory has been derived for the stress analysis of generally anisotropic and heterogeneous elastic bodies of revolution, provided the material structure and loading are axisymmetric. Emphasis has been placed on bodies fabricated by any of the various forms of involute construction. The effects of pore pressure, free expansion strain, body forces, and temperature dependent moduli are incorporated in the model. Owing to the method of formulation, the new theory features the capability to improve solution accuracy by increasing the number of subregions. Solutions for bodies of practical interest will be presented in a subsequent communication.

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